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THE APPLICATION OF THE CRAMER-RAO BOUND TO ESTIMATES OF RADAR RETURN TIME-OF-ARRIVAL FOR SEVERAL TARGET CONFIGURATIONS

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Abstract

In this report, certain applications of the Cramer-Rao bound to problems in radar echo time-of-arrival estimation are presented. These applications encompass point target echos, two-point target echos, and distributed target echos. With respect to point targets and two-point targets, a modified Cramer-Rao bound is derived and the results are substantially tighter than those of the conventional Cramer-Rao bound. In applying the Cramer-Rao bound to distributed targets, a sampled-data formulation is presented and a numerical example corresponding to a wake parameter estimation problem is given to illustrate the results.

A

CONTENTS

1. Introduction

The familiar Cramer-Rao bound on the variance of estimates is particularly useful since it is frequently easy to calculate, and is generally a tight bound for a wide class of estimators when the signal-to-noise ratio is high (References [1]-[4]). Although there are other bounds also tight for small signal-to-noise ratios, they are usually more difficult to obtain (References [1] -[6]). For this reason, the Cramer-Rao bound is often used in radar applications.

In estimating unknown (deterministic or random) parameters from noisy measurements, there often exist nuisance parameters (deterministic or random). In radar applications, the parameter of interest may be the radar return time-of-arrival, or target range, while the nuisance parameters may be signal amplitude and/or phase. Although we often do not care to estimate these nuisance parameters, the estimates of interesting parameters are nevertheless dependent on them. In the usual application of the Cramer-Rao bound, the marginal density of the measurement conditioned on the interesting parameter is used so that the dependence on the nuisance parameter is never explicitly shown. This may sometimes result in optimistic or even trivial bounds. In this report, we obtain a modified bound by treating the nuisance parameters differently. These modified bounds apply to a different, more restricted class of estimators than the conventional Cramer-Rao bound, but are useful in several radar applications where bounds tighter than the ordinary Cramer-Rao bound are obtained.

The application of the Cramer-Rao bound (ordinary and modified) for estimates of parameters of a random signal is, however, usually difficult to obtain. Random signals often occur in practice, for example, the radar return from the wake trailing a high speed re-entry vehicle. Returns of this kind are often thought of consisting of a large number of radar scatterers with their relative amplitude, phase, and location being randomly distributed. They are, therefore, sometimes referred to as distributed targets. It was shown (Reference [2]) that the Cramer-Rao bound for a random signal, or a distributed target, may be obtained via the solution of a rather complicated integral equation. Except for a few special cases, the solution can only be obtained by using approximations. In this note, we consider the calculation of the Cramer-Rao bound for a certain class of random signals via a sampled data approach. The random signal considered is a conditional Gaussian process (Reference [2]) with known time varying mean and covariance functions. Since in many radar applications, the data are often recorded and processed in sampled form, the approach presented in this report is believed to be practical and useful. A numerical example corresponding to a wake parameter estimation problem is given.

This report is organized as follows. The modified Cramer-Rao bound is described in Section 2. Its application to several point targets is given in Section 3. The application of the modified bound to the estimate of time separation of two closely spaced signals is presented in Section 4. In Section 5 we present the sampled data approach to the calculation of the Cramer-Rao bound for distributed targets. A numerical example is included to illustrate the result. Two appendices are attached. In Appendix A the laborious derivation required to obtain the final results of Section 4 is given. There are at least two limiting cases for distributed targets. Under proper conditions, a distributed target converges to a known target or a Rayleigh target. Derivations to explore these relations are given in the Appendix B.

2. A Modified Cramer-Rao Bound

In this section, application of the Cramer-Rao bound to the case wherein there are nuisance parameters is discussed. By treating nuisance parameters in an unusual way, we develop a modified bound. We shall, for simplicity, discuss only the scalar estimation problem although results are easily extended to the multiple parameter estimation problem. Two cases are considered.

CASE 1. Nonrandom Parameter

Let a be an unknown parameter which is to be estimated based on a set of observations $\underline{r}$. Let the joint probability density function (PDF) be known, and equal to

$$p_{\underline{r}}\ (\underline{R}/a) \tag{2.1}$$

The Cramer-Rao bound is a bound on the variance of any unbiased estimator of a: $\hat{a}(\underline{r})$. It is:

$$\sigma_{\hat{a}}^2 \geq \left\{-E\left[\frac{\partial^2}{\partial a^2} \ln p_{\underline{r}}\ (\underline{R}/a)\right]\right\}^{-1} = \left\{E\left[\left(\frac{\partial}{\partial a} \ln p_{\underline{r}}\ (\underline{R}/a)\right)^2\right]\right\}^{-1} = C(a) \tag{2.2}$$

If the observation depends not only on "a" but also on a "nuisance" parameter, θ, then the bound also depends on the value of θ. If θ is random, with a known PDF, then the bound is calculated using

$$p_{\underline{r}}(\underline{R}/a) = \int_{-\infty}^{\infty} p_{\underline{r}}(\underline{R}/\theta,a)\ \rho_\theta(\theta/a)\ d\theta \tag{2.3}$$

The bound then applies to estimators which are unbiased in a global sense, but not necessarily for each value of θ. This leads to the following alternative procedure:

Let $a(\underline{r})$ be an estimator of the parameter a which is unbiased for <u>every</u> value of the nuisance parameter θ. Then

$$\sigma_{\hat{a}}^2 \geq E_\theta\ \{C(a/\theta)\} \tag{2.4}$$

where $C(a/\theta)$ is the Cramer-Rao bound calculated for each value of θ, and E_θ denotes expectation over the random variable θ.

The above result is obvious. For any estimator which is unbiased for all θ, its variance $\sigma_{\hat{a}}^2(\theta)$ for each particular θ is bounded by:

$$\sigma_{\hat{a}}^2(\theta) \geq C(a/\theta) \tag{2.5}$$

where $C(a/\theta)$ is calculated using

$$C(a/\theta) = \left\{-E\left[\frac{\partial^2}{\partial a^2} \ln p_{\underline{r}}(\underline{R}/\theta, a)\right]\right\}^{-1}$$

$$= \left\{ E\left[\left(\frac{\partial}{\partial a} \ln p_{\underline{r}}(\underline{R}/\theta, a)\right)^2\right]\right\}^{-1} \tag{2.6}$$

Since the inequality holds for all θ and since the estimate is unbiased for all θ, it follows that:

$$\sigma_{\hat{a}}^2 = E_\theta \left\{E_{\underline{r}/\theta}\left[\left(\hat{a}(\underline{r})-a\right)^2\right]\right\}$$

$$= E_\theta \left\{\sigma_{\hat{a}}^2(\theta)\right\} \geq E_\theta \left\{ C(a/\theta) \right\} \tag{2.7}$$

CASE 2. Random Parameter

Let a be a random parameter which is to be estimated based on a set of observations $\underline{r}$. Let the joint PDF of $\underline{r}$ and a be known, and denoted by

$$p_{\underline{r},a}(\underline{R},A) \tag{2.8}$$

The Cramer-Rao bound on the variance of any unbiased estimator of a is:

$$\sigma_{\hat{a}}^2 \geq C(a) = \left\{-E_{r,a}\left[\frac{\partial^2}{\partial a^2} \ln p_{\underline{r},a}(\underline{R},A)\right]\right\}^{-1}$$

$$= \left\{E_{r,a}\left[\left(\frac{\partial}{\partial a} \ln p_{\underline{r},a}(\underline{R},A)\right)^2\right]\right\}^{-1} \tag{2.9}$$

If the observation depends also on a parameter θ, then the bound also depends on the value of θ. If θ is random, with a known PDF, then one uses

$$p_{\underline{r},a}(\underline{R},A) = \int_{-\infty}^{\infty} p_{\underline{r},a/\theta}(\underline{R},A/\theta)\, p_\theta(\theta)\, d\theta \tag{2.10}$$

The bound then applies to estimators which are unbiased in a global sense, but not necessarily to estimators which are unbiased for each value of θ. This leads to the following alternate procedure:

Let $\hat{a}(\underline{r})$ be an estimator of the random variable a which is unbiased for <u>every</u> value of the random nuisance parameter θ. Then:

$$\sigma_{\hat{a}}^2 \geq E_\theta\{C(a/\theta)\} \tag{2.11}$$

where

$$C(a/\theta) = \left\{ -E\left[\frac{\partial^2}{\partial a^2} \ln p_{\underline{r},a/\theta} (\underline{R},A/\theta)\right]\right\}^{-1}$$

$$= \left\{ E\left[\left(\frac{\partial}{\partial a} \ln p_{\underline{r},a/\theta} (\underline{R},A/\theta)\right)^2\right]\right\}^{-1} \tag{2.12}$$

The proof follows that for nonrandom parameters.

Discussion:

The difference between our bounds and the standard Cramer-Rao bound on the variance of estimators with a random nuisance parameter is that our bounds apply to estimators which are unbiased for each value of the nuisance parameter whereas the standard bound applies to estimators which need be unbiased only over the ensemble. For instance, in the nonrandom parameter case, if $\hat{a}(\underline{r})$ denotes the estimator, then the standard bound is satisfied if:

$$E_{\underline{r},\theta}(\hat{a}) = \int \hat{a}(\underline{R})\, p_{\underline{r},\theta}(\underline{R},\theta/a)\, d\underline{R}\,d\theta = a \qquad (2.13)$$

whereas our bound is satisfied under the more restricted condition:

$$E_{\underline{r}/\theta}(\hat{a}) = \int \hat{a}(\underline{R})\, p_{\underline{r}/\theta}(\underline{R}/\theta,a)\, d\underline{R} = a \qquad (2.14)$$

$$\text{for all } \theta$$

Certainly, one is not always concerned that an estimator be unbiased for all values of the nuisance parameter. However, there may be many applications where that property is desirable. For instance, in estimating the time-of-arrival of a radar echo of random amplitude, one would certainly like the estimator to be unbiased for all values of signal amplitude. If the time-of-arrival is a nonrandom parameter, this requirement may be unreasonable if the signal amplitude can be too small for reliable detection. However, such an unbiased estimator can be conceived if the time-of-arrival is a random variable, since its a priori distribution provides an unbiased estimate if the signal is not detected. A later set of examples treats this case.

The modified bound which we have presented has a natural interpretation. The quantity $C(a/\theta)$ is the bound which is obtained under the assumption that the nuisance parameter θ is known to the estimator. The modified bound is obtained by simply averaging this quantity over θ. One might expect that a bound obtained under the assumption that the random nuisance parameter is always known to the estimator would not be as tight as one which does not make this assumption. However, this is not always the case, as our examples will demonstrate.

There is another alternative when a nuisance parameter is present. That is, to estimate jointly the values of both the nuisance parameter and the desired parameter. This leads to the Cramer-Rao bound for the multiple parameter case. To evaluate it requires the evaluation of $\frac{1}{2}n(n+1)$ expectations of derivative of logarithms of the joint PDF followed by the inversion of an n by n Fisher Information matrix, where n is the number of parameters estimated. Since, in practice, the estimator actually used is frequently one which estimates nuisance parameters as well as the desired parameter, e.g., a joint maximum likelihood estimator, such a bound may be more representative of the actual situation. Unfortunately, the difficulty in evaluating the bound increases as the number of nuisance parameters increases and may become intractable rather quickly.

3. Point Targets

In this section, we apply the modified Cramer-Rao bound to several point target configurations. Let s(t) denote the low pass transmitted radar signal, the echo from a point target is therefore the time delayed and complex modulated version of s(t). The complex observation of the returned signal is:

$$r(t) = as(t-\tau) + w(t) \tag{3.1}$$

where: s(t) is the complex signal envelope

$$\int_{-\infty}^{\infty} |s(t)|^2 \, dt = 1 \tag{3.2}$$

w(t) is complex white Gaussian noise with spectral density $N_o/2$ (two sided)

a is the complex signal amplitude

τ is the unknown Time-of-arrival

For a fixed known value of a, the well known Cramer-Rao bound for estimators of τ is [3]:

$$\sigma_{\hat{\tau}}^2 (E) \geq C(E) = \frac{1}{2 \frac{E}{N_o} \beta^2} \tag{3.3}$$

where $E = |a|^2$ = energy in the signal

β = signal root-mean-square bandwidth in radians/sec

$$\beta^2 = -\frac{d^2}{d\tau^2}\left[\int_{-\infty}^{\infty} s(t+\tau/2)\, s^*(t-\tau/2)\, dt\right]\Bigg|_{\tau=0} \tag{3.4}$$

For a LFM radar pulse, the relation between β and the instantaneous bandwidth B, in Hz, taking into account the weighting used to reduce sidelobes, is $\beta \approx 1.2$ B.

In practice, the complex signal amplitude a may fluctuate according to some random distribution. Suppose that the complex signal amplitude is random with phase uniformly distributed over $[o, 2\pi]$ and energy distributed according to a PDF p(E). Ideally, we would wish the time-of-arrival estimator to be unbiased for all values of E. Following the discussion on the modified Cramer-Rao bound, a bound on the variance of such an estimator is:

$$\sigma_{\hat{\tau}}^2 \geq C = \frac{N_o}{2\beta^2} \int_o^{\infty} \frac{p(E)}{E}\, dE \tag{3.5}$$

Several target fluctuation models were studied by Swerling [7]. In particular, we consider the Swerling II and IV models.

3.1 Swerling IV Targets

If p(E) corresponds to a Swerling IV ("one-dominant plus Rayleigh") amplitude distribution [7]:

$$P(E) = \frac{4E}{E_o^2} e^{\frac{-2E}{E_o}} \qquad (3.6)$$

E_o = mean signal energy

then

$$\sigma_{\hat{\tau}}^2 \geq C = \frac{1}{\frac{E_o}{N_o} \beta^2} \qquad (3.7)$$

Notice that the above bound is twice as large as that for the known amplitude and phase case.

3.2 Swerling II Targets

If, on the other hand, p(E) corresponds to a Swerling II (Rayleigh) amplitude distribution [7]:

$$p(E) = \frac{1}{E_o} e^{-\frac{E}{E_o}} \qquad (3.8)$$

E_o = mean signal energy

then, for this case in which low-energy signals are more likely than in the previous case, the integral (3.5) for the bound diverges, i.e.,

$$C = \frac{N_o}{2\beta^2 E_o} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{e^{-E/E_o}}{E} \, dE = \infty \tag{3.9}$$

We have therefore shown that no finite variance estimator of (non-random) time-of-arrival, which is also unbiased for all values of signal energy, exists for Rayleigh amplitude distributions. For the same amplitude distribution, the ordinary Cramer-Rao bound exists, and is [2]:

$$\sigma_{\hat{\tau}}^2 \geq \frac{1+\frac{N_o}{E_o}}{2\,\frac{E_o}{N_o}\,\beta^2} = C_o \tag{3.10}$$

One arbitrary remedy to this case is to only estimate target range after a reliable detection of the signal. Another method, which can be rigorously formulated, is to estimate the target range only at the vicinity where the signal is likely to appear. This leads to the following case.

3.3 Swerling II Targets with A Priori Information

Assuming that the time-of-arrival is random with a normal a priori distribution with variance $\sigma_{\hat{\tau}}^2$, the standard Cramer-Rao bound is then:

$$\sigma_{\hat{\tau}}^{2} \geq \left[C_o^{-1} + \frac{1}{\sigma_\tau^{2}} \right]^{-1} \tag{3.11}$$

where C_o is the bound of equation (3.10), i.e., without a priori information about τ. Our modified bound is:

$$\sigma_{\hat{\tau}}^{2} \geq C = \frac{1}{2 \frac{E_o}{N_o} \beta^2} \int_{\epsilon}^{\infty} \frac{e^{-u}}{u + \epsilon} \, du \tag{3.12}$$

where

$$\epsilon = \frac{1}{2 \frac{E_o}{N_o} \beta^2 \sigma_\tau^{2}}$$

Figure (3.1) shows the ratio of the modified bound (3.12) to the standard bound (3.11) as a function of the parameter $\beta\sigma_\tau$. Thus, the resolution of the signal improves relative to the accuracy of the a priori information as the abscissa increases.

So long as the a priori variance is nonzero, the modified bound exists and exceeds the standard bound except for low signal-to-noise ratio and relatively accurate a priori knowledge. We therefore infer that for the Rayleigh channel the modified bound

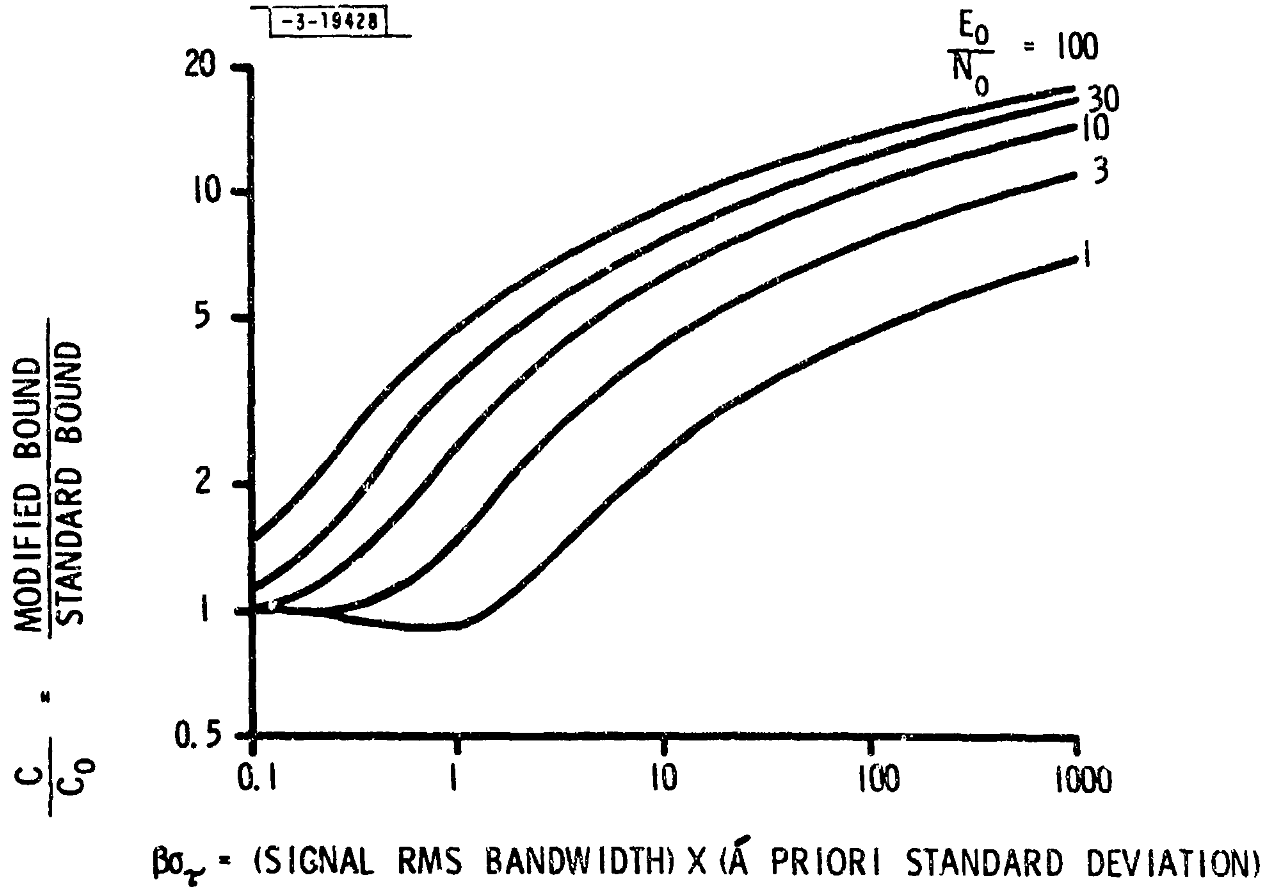


Fig. 3.1. Comparison of modified and standard bounds on variance of time-of-arrival estimates for a Rayleigh channel where time-of-arrival is a random variable.

is usually tighter than the standard bound for estimators of the random time-of-arrival which are unbiased for all signal amplitudes.

4. Two-Point Targets - Length Estimation

Consider the radar application in which two scattering centers are spaced closer than the signal resolution width. This situation occurs when one attempts to estimate the length of a target with only two dominant scatterers. Let the echo from the ith scatterer be denoted as $a_i\, s(t-\tau_1)$. The total received signal may be expressed as

$$r(t) = a_1\, s(t-\tau_1) + a_2\, s(t-\tau_2) + w(t) \tag{4.1}$$

where a_1 and a_2 are the complex signal amplitudes. The parameter to be estimated is the time separation of the targets, $\tau=\tau_1-\tau_2$. For fixed values of $|a_1|$ and $|a_2|$, a conventional way of evaluating the Cramér-Rao bounds leads to the following bound [2].

$$C_o(E_1,E_2) = \frac{1}{2\beta^2}\left(\frac{1}{\frac{E_1}{N_o}} + \frac{1}{\frac{E_2}{N_o}}\right) \tag{4.2}$$

where the $E_i = |a_i|^2$ are energies in the signals. This bound is obtained by assuming that the τ_1 and τ_2 are nonrandom parameters and the $\arg(a_1)$ and $\arg(a_2)$ are random, uniformly distributed on (0.2π), and applies to joint estimators of all four parameters.

It does not include the effect of interference between the signals, hence is tight only if the signals are well resolved.

To find the modified bound one first obtains the standard bound for fixed $|a_1|$ and $|a_2|$ by evaluating and inverting a two by two Fisher information matrix:

$$C(a_1,a_2) = \frac{1}{2\beta^2} \left(\frac{1}{\frac{E_1}{N_o}} + \frac{1}{\frac{E_2}{N_o}} \right) \left[\frac{1+2\left(\frac{\sqrt{E_1E_2/N_o^2}}{E_1/N_o + E_2/N_o}\right)\frac{\ddot{\rho}(t)}{\beta^2} \cos\theta}{\left(1-\frac{\ddot{\rho}(t)}{\beta^2} \cos\theta\right)^2} \right] \tag{4.3}$$

where $\theta = \arg(a_1) - \arg(a_2)$

$$\ddot{\rho}(\tau) = \frac{d^2}{d\tau^2} [\rho(\tau)]$$

$$\rho(\tau) = \int_{\infty}^{\infty} s(t+\tau/2)\ s^*\ (t-\tau/2)\ dt$$

= signal autocorrelation function

Assuming that θ is uniformly distributed in $[0,2\pi]$ and obtaining the modified bound by averaging (4.3) over θ yield

$$C(E_1,E_2) = \frac{\frac{1}{2\beta^2}\left(\frac{1}{\frac{E_1}{N_o}} + \frac{1}{\frac{E_2}{N_o}}\right)}{\sqrt{1 - \left(\frac{\ddot{\rho}(\tau)}{\beta^2}\right)^2}} \tag{4.4}$$

Normalizing the modified bound by the conventional bound, i.e., equation (4.2), yields

$$\frac{C(E_1,E_2)}{C_o(E_1,E_2)} = \frac{1}{\sqrt{1 - \left(\frac{\ddot{\rho}(\tau)}{\beta^2}\right)^2}} \tag{4.5}$$

In order to evaluate the above ratio numerically, a signal model must be chosen. Suppose the signal auto-correlation function is Gaussian-shaped

$$\rho(\tau) = e^{-1/2(\beta\tau)^2} \tag{4.6}$$

Letting $\ell = \beta\tau$

= normalized separation

and evaluating (4.5) yield

$$\frac{C(E_1,E_2)}{C_o(E_1,E_2)} = \frac{1}{\sqrt{1 - (1-\ell^2)^2 e^{-\ell^2}}} \tag{4.7}$$

One can obtain an even tighter bound by treating all four unknowns, i.e., τ_1, τ_2, $\arg(a_1)$ and $\arg(a_2)$ as parameters to be estimated, evaluating the four by four Fisher information matrix, inverting it, and then averaging over $\arg(a_1)$ and $\arg(a_2)$ to obtain the modified bound.

This somewhat laborious procedure is carried out in Appendix A. The result is:

$$C'(E_1,E_2) = \frac{(E_1+E_2)N_o}{2E_1E_2} \quad \frac{1}{2\pi} \int_o^{2\pi} \frac{\psi(o,\tau,\theta)\, d\theta}{[\psi^2(o,\tau,\theta) - \psi^2(\tau,\tau,\theta)\cos^2\theta]} \tag{4.8}$$

where

$$\psi(\tau_1,\tau_2,\theta) = -\left[\ddot{\rho}(\tau_1) + \frac{\rho(\tau_1)\dot{\rho}^2(\tau_2)\,\sin^2\theta}{[1-\rho^2(\tau_2)\cos^2\theta]}\right] \tag{4.9}$$

The above bound normalized by the conventional bound (eq. (4.2)), averaged over θ, and using the Gaussian-shaped autocorrelation function is shown in Figure (4.1). Also shown is the ratio of eq. (4.7) which was more simply obtained by always treating $\arg(a_1)$ and $\arg(a_2)$ as nuisance parameters. Notice that

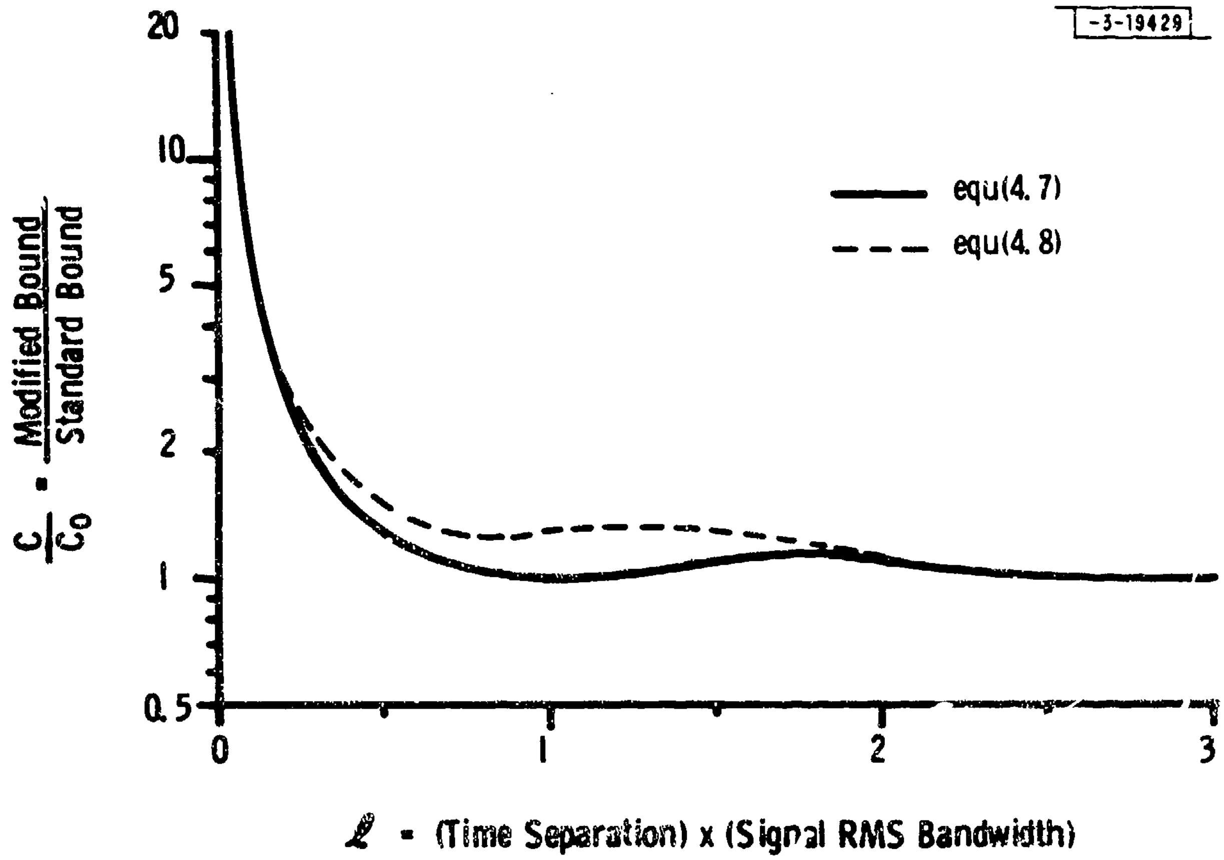


Fig. 4.1. Comparison of modified and standard bounds on variance of estimating the time-separation of two interfering signals.

when the separation is large ($\ell > 3$), these three bounds are identical; while when the separation is small, the modified bounds are much tighter, especially when ℓ is less then unity. The modified bound obtained under the assumption that $\arg(a_1)$ and $\arg(a_2)$ are to be estimated (eq. (4.8)) is only slightly tighter than that obtained under the assumption that they are known to the estimator (eq. (4.7)).

5. Distributed Targets

In this section, we turn our attention to a rather different class of radar returns, namely, the distributed targets. One very commonly known distributed radar echo is the wake trailing a re-entry vehicle. Such signals also occur as a result of disturbances in communication media. References [2] and [8] contain several examples of random signals in radar and communication applications. Figure (5.1) is a simple illustration of random signals contained in radar returns.

Although the basic formulation of the Cramer-Rao bound is the same for various target configurations (Section 2), the explicit solution of the Cramer-Rao bound for estimates of parameters of a random signal is, however, usually difficult to obtain. It was shown (Reference [2]) that the Cramer-Rao bound for a random signal may be obtained via the solution of a rather complicated integral equation. Except for a few special cases, the solution can only be obtained by using approximations. In this note, we consider the

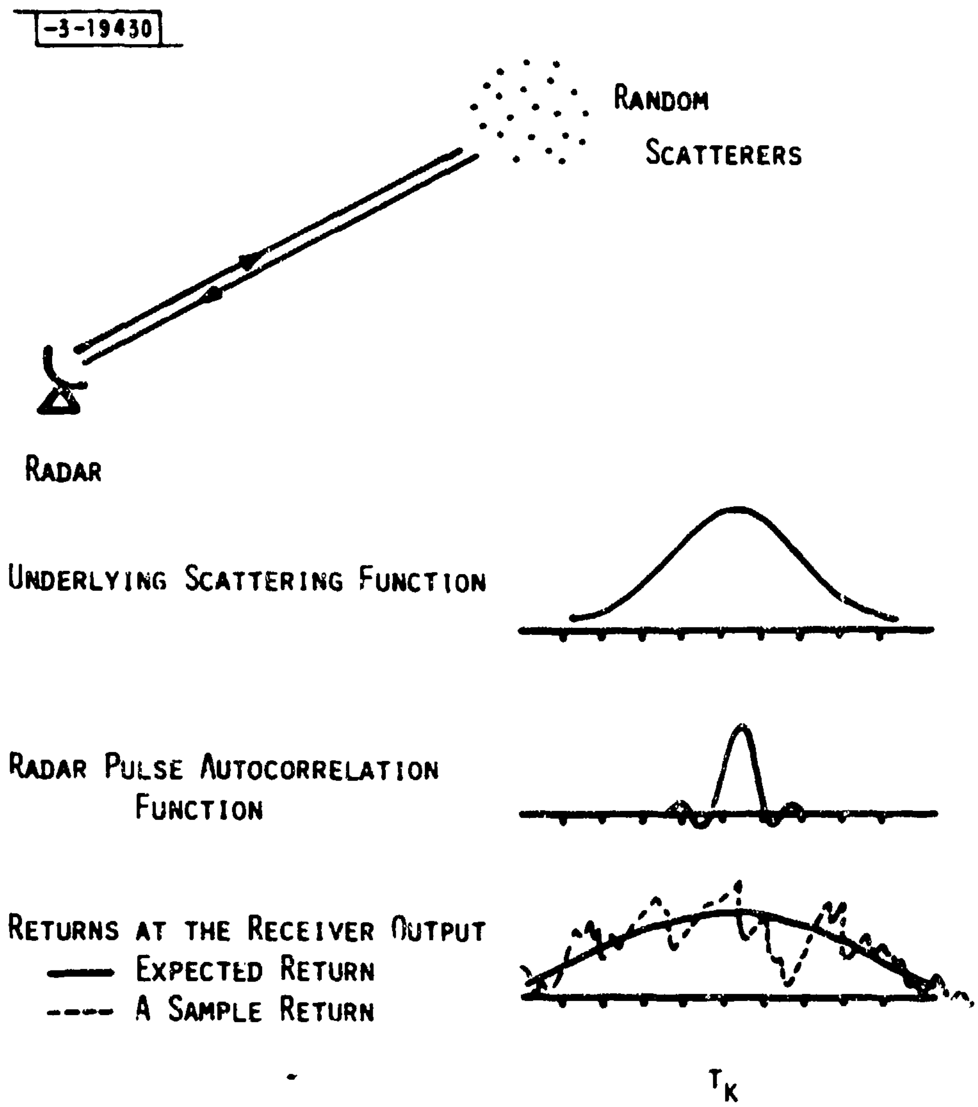


Fig. 5.1. Radar returns from targets with a random scattering function.

calculation of the Cramer-Rao bound for a certain class of random signals via a sampled data approach. The random signal considered is assumed to be a conditional Gaussian process [2] with known time varying mean and covariance functions. Since many radar data are recorded in sampled form, the solution presented here seems to be useful.

We present a derivation of our main result in the Section 5.1. A numerical example is given in the Section 5.2. A deterministic point target and a Rayleigh target are limiting cases of the distributed target. The derivation which shows this relation is presented in the Appendix B.

5.1 Derivation

Since the distributed target is a very different class of target from those discussed in the previous sections, here we re-state the Cramer-Rao bound with a slight inconsistency in notation:

Let "a" denote the parameter to be estimated. The Cramer-Rao bound on the variance of any unbiased estimator of a can be expressed as

$$\sigma_{\hat{a}}^2 \geq C = \left\{ E\left(\left[\frac{\partial}{\partial a} \ln (p_{Y/a}(Y/a))\right]^2 \right) \right\}^{-1}$$

$$= \left\{ -E\left[\frac{\partial^2}{\partial a^2} \ln (p_{Y/a}(Y/a)) \right] \right\}^{-1} \tag{5.1}$$

where $\sigma_{\hat{a}}^2$ = variance of an unbiased estimate of a

Y = measurement process.

A distributed target is characterized by a complex scattering function $h(t,a)$ which is a conditional Gaussian random process with mean

$$E(h(t,a)/a) \stackrel{d}{=} M_h(t,a) \tag{5.2}$$

and covariance function

$$\begin{aligned} E[(h(t,a) - M_h(t,a)) \; (h(u,a) - M_h(u,a))^*/a] \\ = H(t,a) \; \delta(t-u) \end{aligned} \tag{5.3}$$

where "*" denotes the complex conjugate and $\delta(\ )$ is the Dirac delta function. In addition, we assume that the measurement process Y of eq. (5.1) is at the output of a matched filter. Let $s(t)$ denote the transmitted signal, the returned signal before the receiver is

$$\omega(t,a) = \int s(t-x) \; h(x,a) \; dx \tag{5.4}$$

The output of the matched filter is

$$y(t,a) = \int \omega(x,a)\ s^*(x-t)\ dx + \int n(x)\ s^*(x-t)\,dx \qquad (5.5)$$

where n(t) is the receiver noise with Gaussian distribution, zero mean, and one sided spectral density N_o. Introducting the signal autocorrelation function

$$\rho(t-\tau) = \int s(x-\tau)\ s^*(x-t)\,dx \qquad (5.6)$$

Equation (5.5) can be rewritten as

$$y(t,a) = \int \rho(t-x)\ h(x,a)\,dx + \int n(x)\ s^*(x-t)\,dx \quad (5.7)$$

The output process y(t,a) is a conditionally Gaussian random process with mean

$$E(y(t,a)/a) = M_y(t,a) = \int \rho(t-x)\ M_h\ (x,a)\,dx \qquad (5.8)$$

and covariance function

$$S(t_1,t_2,;a) = \int \rho(t_1-x)\ \rho(t_2-x)\ H(x,a)\,dx + N_o\rho(t_1-t_2) \qquad (5.9)$$

In many radar and communication applications, the data at the matched filter output are often recorded and processed in sampled form. Let Y_{1i} and Y_{2i} denote the data vectors representing the sampled outputs from inphase and quadrature channels from the ith set of measurements (or, a set of sampled data of the ith radar pulse). Furthermore, it is assumed that the significant portion of the signal energy is included in the data vector. The covariance of Y_{ij} is

$$R = \begin{bmatrix} S(t_1,t_1;a) & S(t_1,t_2;a) \ldots & S(t_1,t_M;a) \\ S(t_2,t_1;a) & S(t_2,t_2;a) \ldots & S(t_2,t_M;a) \\ \vdots & \vdots & \vdots \\ S(t_M,t_1;a) & S(t_M,t_2;a) \ldots & S(t_M,t_M;a) \end{bmatrix} \quad (5.10)$$

; for all i,j

where M is the number of samples in the data vector. The conditional density for total N pulses can be expressed as

$$P_{y/a}(Y/a) = \frac{1}{(2\pi |R|)^N} \left\{ \exp -1/2 \sum_{i=1}^{2} \sum_{j=1}^{N} (Y_{ij}-M_y)^T R^{-1} (Y_{ij}-M_y) \right\} \quad (5.11)$$

where M_y is the vector with samples $M_y(t_i,a)$ i=1,..,M. Differentiating the natural log of $P_{y/a}(Y/a)$ with respect to a yields

$$\frac{\partial}{\partial a} [\ln(p_{Y/a}(Y/a))] = \frac{-1}{2} \sum_{i=1}^{2} \sum_{j=1}^{N} \left\{ (Y_{ij}-M_Y)^T \frac{\partial (R^{-1})}{\partial a} (Y_{ij}-M_Y) \right. \tag{5.12}$$

$$\left. - 2(Y_{ij}-M_Y)^T R^{-1} \left(\frac{\partial M_Y}{\partial a}\right) \right\}$$

Squaring (5.12) and taking expectation yield

$$E\left\{\left(\frac{\partial}{\partial a} [\ln(p_{Y/a}(Y/a))] \right)^2\right\} \tag{5.13}$$

$$= N\, Tr\left[\left(R^{-1} \frac{\partial R}{\partial a}\right)^2 \right] + 2N \left(\frac{\partial M_Y}{\partial a}\right)^T R^{-1} \left(\frac{\partial M_Y}{\partial a}\right)$$

where Tr[ ] denotes the trace of the enclosed matrix. To obtain the above results, the following assumptions and matrix identities have been used.

Assumptions

1) The data from the inphase and quadrature channels are independent
2) For $j \neq k$, Y_{ij} and Y_{ik} are independent

Matrix Indentities

1) $E\left\{(Y_{ij}-M_Y) \quad [(Y_{ij}-M_Y)^T \quad R^{-1} \quad (Y_{ij}-M_Y)]\right\} = \underline{0}$
2) $E\left[Y_{ij}^T K Y_{ij}^T K Y_{ij}\right] = 2Tr[KRKR] + Tr^2[KR]$

where $K = \dfrac{\partial R^{-1}}{\partial a}$

$\underline{0} \triangleq$ a zero vector

3) $Tr[KR] = 0$

The final result for the Cramer-Rao bound is

$$\sigma_{\hat{A}}^2 \geq C = \frac{1}{N} \left\{ Tr \left[\left(R^{-1} \frac{\partial R}{\partial a} \right)^2 \right] + 2 \left(\frac{\partial M_y}{\partial a} \right)^T R^{-1} \left(\frac{\partial M_y}{\partial a} \right) \right\}^{-1} \tag{5.14}$$

This equation is the main result for the distributed target case.

We make the following remarks:

1) This formula is an approximate solution of the exact formulation in Reference [2]. It is a valid approximation if the data vector contains the significant portion of the signal energy. This requirement can be easily satisfied in radar applications.

2) Since the data vectors are assumed independent, the above result applies to any unbiased estimators which process N pulses (or data sets) incoherently.

3) In most communication and radar signal processing applications, the fading channel/fluctuating targets are modeled by using a scattering function specified by $M_h(t,a)$ and $H(t,a)$. The Cramer-Rao bound is computed by first evaluating equations (5.8) and (5.9) to obtain $M_y(t,a)$ and $S(t_i;t_j;a)$. This step may require numerical integration.

4) The above result is explicitly shown for time domain data. Due to the duality property between time and frequency, it

may be applied to frequency domain data (e.g., estimating the Doppler spread of a fading signal) in a straightforward manner.

5) This result can be extended to the multiple parameter case by computing each terms of the Fisher information matrix (Reference [1]) with the same approach as above.

5.2 Example

In this section, we consider an example which applies to targets with zero mean and Gaussian-shaped covariance functions, i.e.,

$$M_h(t,a) \equiv M_y(t,a) \equiv 0. \quad \text{for all } t \tag{5.15}$$

and

$$H(t,a) = E \frac{1}{\sqrt{2\pi}\,\sigma_h} \exp\left\{-1/2 \frac{(t-a)^2}{\sigma_h^2}\right\} \tag{5.16}$$

where E is the signal energy. Notice that the parameter to be estimated, a, is the center of H(t,a). Assuming that the radar signal is a LFM pulse, the signal autocorrelation function of a Hamming weighted LFM pulse can be approximated by

$$\rho(t) = e^{-\frac{t^2}{2}(1.226B)^2} \tag{5.17}$$

where B is the LFM bandwidth in Hz. Notice that with the above assumptions, equation (5.9) can be evaluated analytically. Define the following peak signal-to-noise ratio (S/N) at the receiver output by using equation (5.9).

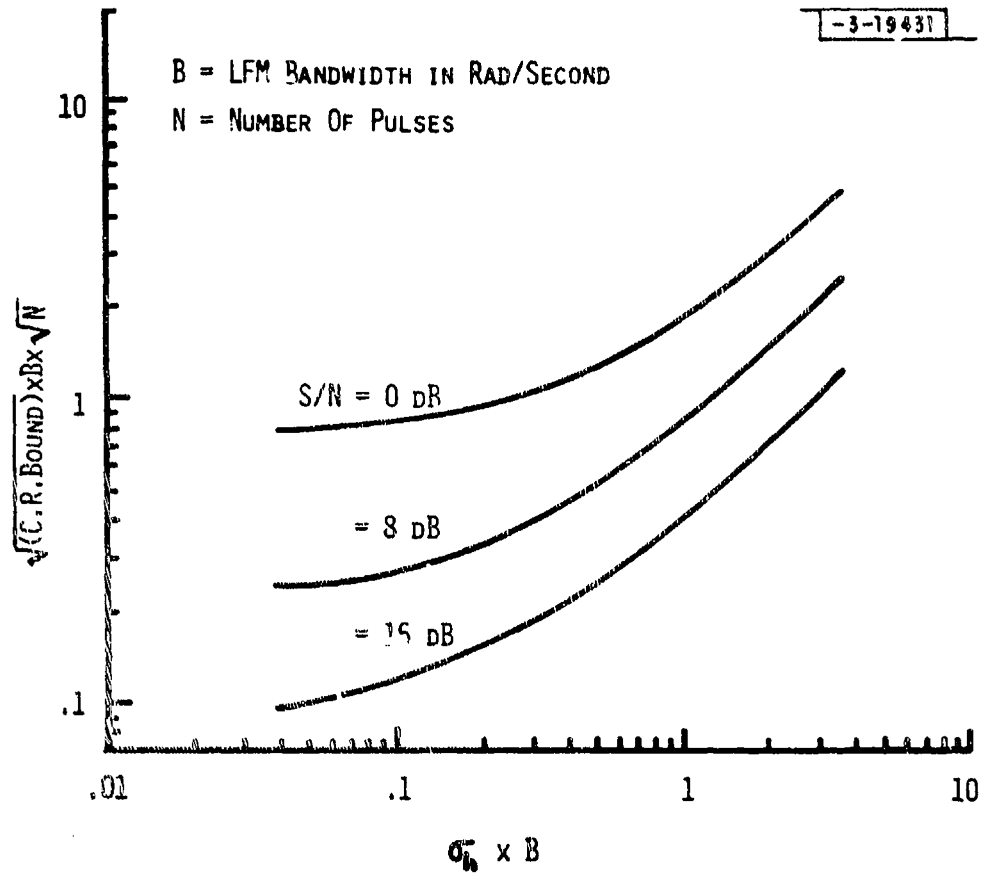
$$S/N = \frac{1}{N_o} \int_{-\infty}^{\infty} |f(x)|^2 \; H(x,a)\,dx \qquad (5.18)$$

Using the above assumptions, the Cramer-Rao bound normalized to the inverse of $B\sqrt{N}$ is evaluated with respect to the target second central moment (σ_h) normalized to the inverse of B where N is the number of pulses. The results for S/N = 0,8, and 16 dB's are shown in Figure (5.2). Notice that the estimate standard deviation degrades rapidly for random targets with large time-spread (σ_h). When the time-spread becomes small, the Cramer-Rao bound is asymptotic to that for a Rayleigh target.

The above result can be extended to parameter estimates of a RV wake. Results of Figure 5.2 correspond to the time-of-arrival estimation of wakes with Gaussian shaped RCS. For wakes with any other shapes, one simply replaces H(t,a) by the underlying model corresponding to that shape where "a" is the wake parameter to be estimated.

6. Summary and Conclusions

Certain applications of the Cramer-Rao bound to problems in



For Rayleigh Targets: $\sqrt{(C.R.B.)} \times B \times \sqrt{N}$ = .78 for S/N = 0 dB
= .236 = 8 dB
= .088 = 16 dB

Fig. 5.2. The Cramer-Rao bound for estimating the mean of a time-spread target.

radar echo time-of-arrival estimation have been presented. These applications encompass point target echos, two-point target echos, and distributed target echos.

With respect to point targets, application of the modified bound on the variance of time-of-arrival estimates, valid for estimators which are unbiased for every value of signal amplitude, shows the following:

a. the bound for Swerling IV targets is twice that for nonfluctuating targets having the same mean signal energy.

b. No such estimator has finite variance for Swerling II targets unless there is a priori information regarding the time-of-arrival. When such information is available, the resulting modified bound, equation (3.12), is usually tighter than the standard bound commonly used.

When applied to the problem of estimating the apparent extent of a two-point target, the standard Cramer-Rao bound, equation (4.2), is independent of the target extent; whereas the modified bound, equation (4.4), does display the effect of interference between the two point scatterers and is uniformly tighter than the standard bound. Results shown in Figure 4.1 indicate that the effective resolution of a signal, for this purpose, is about 30% to 50% of its reciprocal RMS bandwidth in radians/sec. This corresponds to 25% to 40% of the reciprocal instantaneous bandwidth for a LFM pulse.

In applying the Cramer-Rao bound to estimates of the parameters of echos from distributed targets, a sampled-data formulation was used. The general result, equation (5.14), for the scalar case was applied to the case wherein the target is random having a scattering function which is a Gaussian-shaped function of range. Results of this analysis, given in Figure 5.2, show the degradation of the time-of-arrival estimate as the time-spread of the target echo increases.

Although our discussion has centered on the analysis of accuracy of time-of-arrival estimates, the results are equally applicable to problems in estimating Doppler shift, recognizing the time-frequency duality. Thus, all of our analyses can be made to apply to Doppler shift estimation by simply interchanging frequency and time variables. Similarly, the results are applicable to angle-of-arrival estimation problems for a linear aperture by appropriately reinterpreting the time variable as off-boresight angle and the frequency variable as distance along the aperture, measured in wavelengths.

APPENDIX A

Derivation of Equation (4.8)

Equation (4.8) was obtained by evaluating and inverting the Fisher Information matrix by treating τ_1, τ_2, $\arg(a_1)$ and $\arg(a_2)$ as parameters. Restating the received signal

$$r(t) = a_1 s(t-\tau_1) + a_2 s(t-\tau_2) + w(t) \tag{B.1}$$

The log of the conditional density function is

$$\ln p_r = \frac{-1}{N_o} \int \left| r(t) - a_1 s(t-\tau_1) - a_2 s(t-\tau_2) \right|^2 dt \tag{B.2}$$

Evaluating partial derivatives of $\ln p_r$ and taking expectations yield

$$E\left(\frac{\partial^2 \ln p_r}{\partial \tau_i^2}\right) = 2|a_i|^2 \ \ddot{\rho}(0)$$

$$E\left(\frac{\partial^2 \ln p_r}{\partial \tau_i \partial \tau_j}\right) = -2|a_i| \ |a_j| \ \ddot{\rho}(\tau) \cos\theta \quad ; \text{ for } i \neq j$$

$$E\left(\frac{\partial^2 \ln p_r}{\partial \tau_i \ \partial \theta_i}\right) = 0$$

$$E\left(\frac{\partial^2 \ln p_r}{\partial \tau_i \partial \theta_j}\right) = -2|a_i||a_j|\, \dot{\rho} \sin\theta \qquad ;\ \text{for } i \neq j$$

$$E\left(\frac{\partial^2 \ln p_r}{\partial \theta_i^{\,2}}\right) = -2|a_i|^2$$

$$E\left(\frac{\partial^2 \ln p_r}{\partial \theta_i \partial \theta_j}\right) = -2|a_i||a_j|\, \rho(\tau) \cos\theta \qquad ;\ \text{for } i \neq j$$

where $\theta = \arg(a_1) - \arg(a_2)$

$\tau = \tau_1 - \tau_2$

$\rho(\tau)$ = signal autocorrelation function

$= \int s(t)\, s^*(t-\tau)\, dt$

$\dot{\rho} = \frac{d\rho}{d\tau}$

$\ddot{\rho} = \frac{d^2\rho}{d\tau^2}$

Using the above results, the Fisher Information matrix is:

$$
F = \frac{2}{N_o} \begin{bmatrix} -|a_1|^2 \ddot{\rho}(o) & |a_1||a_2|\ddot{\rho}(\tau)\cos\theta & 0 & |a_1||a_2|\dot{\rho}(\tau)\sin\theta \\ |a_1||a_2|\ddot{\rho}(\tau)\cos\theta & -|a_2|^2\ddot{\rho}(o) & |a_1||a_2|\dot{\rho}(\tau)\sin\theta & 0 \\ 0 & |a_1||a_2|\dot{\rho}(\tau)\sin\theta & |a_1|^2 & |a_1||a_2|\rho(\tau)\cos\theta \\ |a_1||a_2|\dot{\rho}(\tau)\sin\theta & 0 & |a_1||a_2|\rho(\tau)\cos\theta & |a_2|^2 \end{bmatrix} \tag{B.3}
$$

Since we are only interested in evaluating the bound on τ, one need not invert the complete matrix. Instead, we will apply the following matrix identity to find the inverse corresponding to the upper 2x2 portion of F.

Matrix Identity

Given

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where A_{ij} and B_{ij} are partitions of A and B, respectively, then

$$B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}$$

Applying the above identity to matrix F yields

$$F_{11} - F_{12}F_{22}^{-1}F_{21}$$

$$= \frac{2}{N_o}\begin{bmatrix} |a_1|^2\left(-\ddot{\rho}(0) - \dfrac{\dot{\rho}^2(\tau)\ \sin^2\theta}{1-\rho^2(\tau)\cos^2\theta}\right) & |a_1||a_2|\left(\ddot{\rho}(\tau) + \dfrac{\rho(\tau)\dot{\rho}^2(\tau)\sin^2\theta}{1-\rho^2(\tau)\cos^2\theta}\right)\cos\theta \\ |a_1||a_2|\left(\ddot{\rho}(\tau) + \dfrac{\rho(\tau)\dot{\rho}^2(\tau)\sin^2\theta}{1-\rho^2(\tau)\cos^2\theta}\right)\cos\theta & |a_2|^2\left(-\ddot{\rho}(0) - \dfrac{\dot{\rho}^2(\tau)\sin^2\theta}{1-\rho^2(\tau)\cos^2\theta}\right) \end{bmatrix} \tag{B.4}$$

Let

$$\psi(\tau_1,\tau_2,\theta) = -\left(\ddot{\rho}(\tau_1) + \frac{\rho(\tau_1)\dot{\rho}^2(\tau_2)\ \sin^2\theta}{1-\rho^2(\tau_2)\ \cos^2\theta}\right) \tag{B.5}$$

Equation (B.4) becomes

$$F_{11} - F_{12}\ F_{22}^{-1}\ F_{21}$$

$$= \frac{2}{N_o}\begin{bmatrix} |a_1|^2\ \psi(0,\tau,\theta) & |a_1||a_2|\psi(\tau,\tau,\theta)\ \cos\theta \\ |a_1||a_2|\ \psi(\tau,\tau,\theta)\ \cos\theta & |a_2|^2\ \psi(0,\tau,\theta) \end{bmatrix} \tag{B.6}$$

Its inverse is

$$(F_{11}-F_{12}F_{22}^{-1}F_{21})^{-1}$$

$$= \frac{N_o \begin{bmatrix} |a_2|^2\psi(0,\tau,\theta) & -|a_1||a_2|\psi(\tau,\tau,\theta)\cos\theta \\ -|a_1||a_2|\psi(\tau,\tau,\theta)\cos\theta & |a_1|^2\psi(0,\tau,\theta) \end{bmatrix}}{2|a_1|^2|a_2|^2\,(\psi^2(0,\tau,\theta) - \psi^2(\tau,\tau,\theta)\cos\theta)} \tag{B.7}$$

The bound on σ_τ^2 is therefore

$$\sigma_\tau^2 \geq C'(E_1,E_2,\theta)$$

$$= \frac{N_o}{2} \quad \frac{(E_1+E_2)\;\psi(0,\tau,\theta) + 2\sqrt{E_1E_2}\;\psi(\tau,\tau,\theta)\cos\theta}{E_1E_2\;[\psi^2(o,\tau,\theta) - \psi^2(\tau,\tau,\theta)\cos^2\theta\;]} \tag{B.8}$$

where $E_1=|a_1|^2$ and $E_2=|a_2|^2$ were used. The modified bound is obtained by averaging $C'(E_1,E_2,\theta)$ over θ, i.e.,

$$C'(E_1,E_2) = \frac{1}{2\pi}\int_o^{2\pi} C'(E_1,E_2,\theta)\;d\theta \tag{B.9}$$

It however, can be shown that

$$\int_0^{2\pi} \frac{\psi(\tau,\tau,\theta) \quad \cos\theta}{[\psi^2(0,\tau,\theta) - \psi^2(\tau,\tau,\theta) \cos^2\theta]} d\theta=0 \tag{B.10}$$

because

$$\frac{\psi(\tau,\tau,\theta)}{[\psi^2(o,\tau,\theta) - \psi^2(\tau,\tau,\theta) \cos^2\theta]}$$

is an even and positive function of θ. Using this property, equation (B.9) becomes

$$C'(E_1,E_2) = \frac{N_o}{2}\left(\frac{E_1+E_2}{E_1E_2}\right)\frac{1}{2\pi}\int_0^{2\pi} \frac{\psi(0,\tau,\theta)\ d\theta}{[\psi^2(0,\tau,\theta) - \psi^2(\tau,\tau,\theta) \cos^2 \theta]} \tag{B.11}$$

This is equation (4.8).

APPENDIX B

Limiting Cases of Distributed Targets

In this appendix, we prove the convergence of the distributed targets to two limiting cases, namely, a target with known amplitude and phase and a Rayleigh target.

B.1 A Target with Known Amplitude and Phase

Assume that the target has a deterministic scattering function with known amplitude and a denotes the time-of-arrival of the returned signal. This implies

$$M_h(t,a) = \sqrt{E}\ \delta\ (t-a) \tag{B.1}$$

and

$$H(t,a) = 0 \tag{B.2}$$

where E is the signal energy. Using the above in (5.8) yields

$$M_Y(t,a) = \sqrt{E}\ \rho(t-a) \tag{B.3}$$

Substituting the above in (5.14) yields the following result for noncoherently integrated N pulses.

$$C(\text{point target}) = \frac{1}{N} \cdot \frac{1}{\frac{2E}{N_o}(\dot{M}_Y{}^T R^{-1}\dot{M}_Y)} \qquad \text{(B.4)}$$

where $\dot{M}_Y = \frac{\partial M_Y}{\partial a}$. When the number of samples in M_Y is large, then

$$\dot{M}_Y{}^T R^{-1} \dot{M}_Y \longrightarrow \beta^2 = (\text{signal r.m.s. bandwidth in rad/sec})^2 \qquad \text{(B.5)}$$

The above bound converges to the Cramer-Rao bound for a known target (Reference [3] or Section 3 above).

B.2 A Rayleigh Target

A Rayleigh target satisfies the following properties

$$M_h(t,a) = 0 \qquad \text{(B.6)}$$

$$H(t,a) = E\ \delta(t-a) \qquad \text{(B.7)}$$

where E is the target energy. The Cramer-Rao bound then becomes

$$C(\text{Rayleigh Target}) = \frac{1}{N} \ \mathrm{Tr} \left[(R^{-1}\dot{R})^2 \right] \qquad (B.8)$$

where $\dot{R} = -E(\dot{\gamma}\ \gamma^T + \gamma\ \dot{\gamma}^T)$

$$R = \Gamma + N_o P$$

$$\Gamma = E\ \gamma\ \ \gamma^T$$

$$P = [\rho(t_i - t_j)]$$

$$\gamma = \begin{bmatrix} \rho(t_1 - a) \\ \rho(t_2 - a) \\ \vdots \\ \rho(t_M - a) \end{bmatrix}$$

$$\gamma = \frac{d\dot{\gamma}}{da}$$

The inverse of R can be expressed as follows by a simple application of the matrix inversion lemma.

$$R^{-1} = \frac{1}{N_o} \left[P^{-1} - P^{-1}\gamma\ (\gamma^T P^{-1}\gamma + N_o/E)^{-1}\ \gamma^T P^{-1} \right] \quad (B.9)$$

We will restrict the number of samples, M, to be an odd integer. When M is even, the modification to the arguments to follow is straightforward. From the definitions above, the $(\frac{M+1}{2})$th row (and column) of P is γ. Other rows/columns of P are just multiple shifts of γ. In addition, the center element of γ is unity and that of $\dot{\gamma}$ is zero. Let q denote a column vector with all elements equal to zero except the $(\frac{M+1}{2})$th element which is equal to one. Then

$$\gamma = Pq \qquad (B.10)$$

or

$$q = P^{-1}\gamma \qquad (B.11)$$

Using the above transformation in (B.9) and after some manipulations one obtains

$$R^{-1} = \frac{1}{N_o}\left[P^{-1} - U\right] \qquad (B.12)$$

$$R^{-1}\dot{R} = \frac{-E}{N_o}\left[P^{-1}(\dot{\gamma}\,\gamma^T + \gamma\,\dot{\gamma}^T) - U(\dot{\gamma}\,\gamma^T + \gamma\,\gamma^T)\right] \qquad (B.13)$$

where U is a (MxM) matrix with all the elements zero except the center element which is equal to $1/(1+N_o/E)$.

Squaring (B.13) and after some manipulations one obtains

$$C(\text{Rayleigh Target}) = \frac{1}{N}\left(\frac{2(E/N_o)^2}{(E/N_o)+1}\right)^{-1} \frac{1}{\dot{\gamma}\, P^{-1} \dot{\gamma}} \qquad (B.14)$$

When the number of samples is large, then

$$\dot{\gamma}^{\,T} P^{-1}\, \dot{\gamma} \longrightarrow \beta^2 \qquad (B.15)$$

The above bound converges to the Cramer-Rao bound for a Rayleigh target (Reference [2] or Section 3 above).

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In this report, certain applications of the Cramer-Rao bound to problems in radar echo time-of-arrival estimation are presented. These applications encompass point target echos, two-point target echos, and distributed target echos. With respect to point targets and two-point targets, a modified Cramer-Rao bound is derived and the results are substantially tighter than those of the conventional Cramer-Rao bound. In applying the Cramer-Rao bound to distributed targets, a sampled-data formulation is presented and a numerical example corresponding to a wake parameter estimation problem is given to illustrate the results.

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